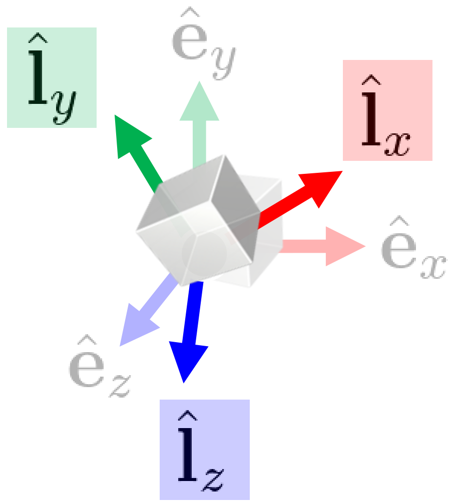
Properties of Rotation Matrix



$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- 1a. The dot product of any row or column with itself equals one
  - Because they are unit vectors

$$\|\hat{\mathbf{i}}_x\|^2 = \|\hat{\mathbf{i}}_y\|^2 = \|\hat{\mathbf{i}}_z\|^2 = 1$$

$$(\|v\| = \sqrt{v \cdot v} = x^2 + y^2 + z^2)$$

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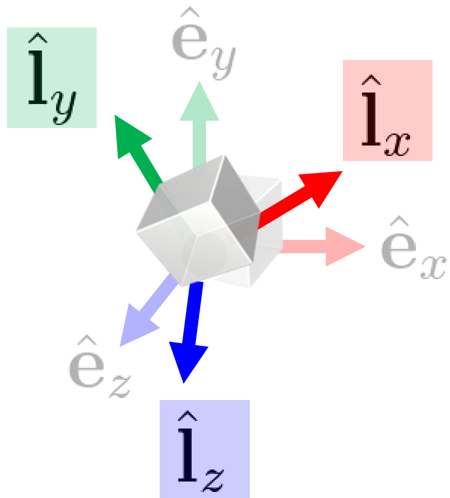


$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- 1b. The dot product of any row with any other row equals zero
- 1c. The dot product of any column with any other column equals zero
  - Because they are perpendicular to each other

$$\hat{\imath}_x \cdot \hat{\imath}_y = \hat{\imath}_y \cdot \hat{\imath}_z = \hat{\imath}_x \cdot \hat{\imath}_z = 0$$

Properties of Rotation Matrix

- From the property 1a, 1b, 1c,

$$\boxed{\textbf{1. } \mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}} \left(\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

- A matrix having this property is called an **orthogonal matrix**
  - So, a rotation matrix is an orthogonal matrix
  - But it has one more property; its determinant is 1

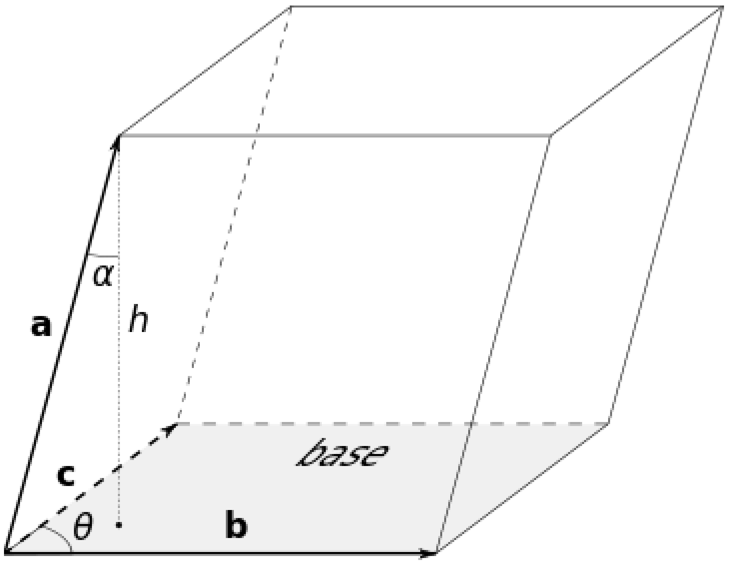
Determinant of 3x3 Matrix

- There are several ways to calculate a matrix determinant
- For **3x3 matrices**, one can use **scalar triple product**

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad (\ \det(A^{\mathrm{T}}) = \det(A)\ )$$

$|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ = the volume of the parallelepiped spanned by **a**, **b**, and **c**

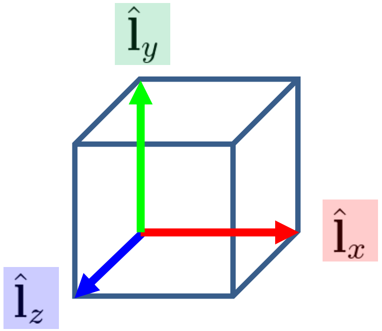
See more: https://mathinsight.org/scalar_triple_product



Determinant of 3x3 Orthogonal Matrix

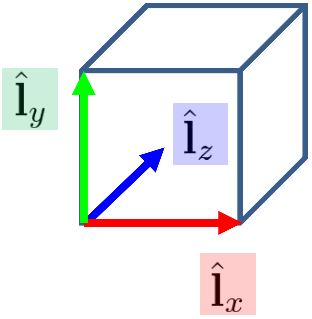
- If **Q** is an orthogonal matrix, det(**Q**) = +1 or -1
  - proof) $1 = \det(I) = \det\left(Q^{\mathrm{T}} Q\right) = \det\left(Q^{\mathrm{T}}\right) \det(Q) = \left(\det(Q)\right)^2.$

- det(**Q**) = +1 → rotation



$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- det(**Q**) = -1 → (rotated) reflection
  - Meaning that one axis is "flipped"



Properties of Rotation Matrix

$$\boxed{\textbf{2.}\quad \det(\mathbf{R}) = 1}$$

- To sum up, a rotation matrix is an **orthogonal matrix with determinant 1**
  - Sometimes it is called *special orthogonal matrix*
  - A set of rotation matrices of size 3 forms a *special orthogonal group, SO(3)*